

GLOBAL  $G$ -MANIFOLD REDUCTIONS AND RESOLUTIONS

KARSTEN GROVE\* AND CATHERINE SEARLE\*\*

*Dedicated to the memory of Alfred Gray*

ABSTRACT. The purpose of this note is to exhibit some simple and basic constructions for smooth compact transformation groups, and some of their most immediate applications to geometry.

It is well known that, if  $G$  is a compact Lie group acting smoothly on a manifold  $M$  with only one orbit type, then the orbit space  $M/G$  is a manifold, and the orbit map  $\pi : M \rightarrow M/G$  is a locally trivial bundle with fiber  $G/H$ , the typical  $G$ -orbit in  $M$ . Moreover, the normalizer  $N(H)$  acts on the fixed point set,  $M^H \subset M$  of  $H$  in  $M$ , with  $H$  as the ineffective kernel, and  $M^H \rightarrow M^H/N(H) = M/G$  is a principal  $N(H)/H$ -bundle. Its associated  $G/H$ -bundle is  $\pi$ . In particular, if we set  ${}_cM = M^H$  and  ${}_cG = N(H)/H$  then  $M = {}_cM \times_{{}_cG} G/H$ . This shows that we can recover  $M$  as a  $G$ -manifold completely from the  ${}_cG$ -manifold  ${}_cM$  and  $H \subset G$ . We will refer to  ${}_cM$  as the *core* of  $(M, G)$  and to  ${}_cG$  as its *core group*.

In general, when the action  $G \times M \rightarrow M$  has more than one orbit type no such simplification exists. It turns out, however, that if  $H$  is a principal isotropy group, then the closure,  ${}_cM$  of the core  ${}_c(M_o) = (M_o)^H$  of the regular part  $M_o$  of  $M$  (all principal orbits) is a smooth  ${}_cG$ -manifold which contains considerable information about  $(M, G)$  (see Proposition 1.2 and Proposition 1.4). This was already indicated in [2] and used in [17]. The unpublished manuscript, [16] by T. Skjeldbred and E. Straume is devoted to the basic investigation of this core,  ${}_cM$  of  $(M, G)$  referred to as the *reduction* by them. Given its importance, we have included complete proofs of this basic material. Our proofs are different from those of [16], in that we use Riemannian geometric tools from the outset.

We use the core to obtain restrictions on positively curved  $G$ -manifolds. In particular, we obtain an extension of a fixed point lemma (cf. Corollary 1.8), used in our systematic investigation of symmetry groups of positively curved manifolds in [10].

As another application of the core, we associate to any smooth  $G$ -manifold,  $M$  with  $G$  a compact Lie group another  $G$ -manifold  ${}^rM$  and a smooth surjective  $G$ -map  $f : {}^rM \rightarrow M$ . The orbit space of  ${}^rM$  is the same as that of  $M$ , but  ${}^rM$  is less singular than  $M$  in the sense that corresponding orbits have smaller isotropy groups (Theorem 2.1). For this reason, we think of  ${}^rM$  as a (partial) *resolution* of  $M$ . In contrast to other regularizations of group actions, e.g., *blow-up* along invariant submanifolds as in [20], our construction is completely global in nature. In terms of the core,  ${}^rM = {}_cM \times_{{}_cG} G/H$  fibers over  $G/N(H)$  with fiber  $G/H$ . - A geometric feature of the resolution construction is that it preserved the class of manifolds with non-negative curvature (2.10). In particular, new manifolds of non-negative curvature may possible be constructed by this method. We also point out natural problems and conjectures related to the constructions in this note.

---

\* Supported in part by a grant from the National Science Foundation and by the Danish National Research Council.

\*\* Supported in part by CONaCYT grant no, 28491E..

1. THE CORE OF A  $G$ -MANIFOLD

Since we only consider smooth compact transformation groups, throughout we may as well assume that each transformation is an isometry relative to a fixed (auxiliary) Riemannian metric.

We first recall some well known facts (cf. e.g. [5]) and establish notation. - Throughout,  $M$  will denote a closed, connected Riemannian manifold, and  $G$  a compact Lie group which acts isometrically and effectively on  $M$ . For  $p \in M$ ,  $G_p = \{g \in G \mid gp = p\}$  is the *isotropy group* of  $G$  at  $p$ , and  $Gp = \{gp \mid g \in G\} \simeq G/G_p$  is the orbit through  $p$ . For any (closed) subgroup  $L \subset G$ ,  $M^L = \{p \in M \mid Lp = p\}$  will denote the fixed point set of  $L$  in  $M$ .  $M^L$  is a finite union of closed totally geodesic submanifolds of  $M$ .

We endow the orbit space,  $M/G$  with the so-called *orbital metric*, i.e., if  $\pi : M \rightarrow M/G$  is the quotient map, then the distance between  $\pi(p)$  and  $\pi(q)$  is the Riemannian distance in  $M$  between the orbits  $Gp$  and  $Gq$ . With this metric,  $\pi$  is a *submetry*, i.e., for any  $p \in M$  and any  $r > 0$ , the  $r$ -ball around  $p$ ,  $B(p, r)$  is mapped onto the  $r$ -ball,  $B(\pi(p), r)$  around  $\pi(p)$ . - On the regular part  $M_o \subset M$  consisting of all principal orbits, the restriction  $\pi : M_o \rightarrow M_o/G$  is a Riemannian submersion and a locally trivial bundle map with fiber  $G/H$ , where  $H = G_{p_o}$  for some  $p_o \in M_o$ . If the principal isotropy type ( $H$ ) is trivial, the bundle  $\pi : M_o \rightarrow M_o/G$  is a principal  $G$ -bundle.

Fix a principal isotropy group  $H$ , and assume from now on that  $H \neq \{1\}$ . The normalizer,  $N(H)$  of  $H$  in  $G$  clearly acts on  $M_o^H = M_o \cap M^H$  with  $H$  as ineffective kernel. Moreover, each principal  $G$ -orbit,  $G/H$  intersects  $M_o^H$  in an  $N(H)$ -orbit,  $(G/H)^H \simeq N(H)/H =: {}_cG$ , and the inclusion  $M_o^H \subset M_o$  induces an isometry  $M_o^H/{}_cG \simeq M_o/G$ . The induced action of the *core group*,  ${}_cG$  on  $M_o^H$  is free, and

$$(1.1) \quad M_o^H \rightarrow M_o^H/{}_cG \simeq M_o/G$$

is the principal bundle for  $M_o \rightarrow M_o/G$ . Indeed, the  $G$ -action

$$(1.2) \quad G \times M_o^H \times G/H \rightarrow M_o^H \times G/H, (g, (x, [g'])) \rightarrow (x, [gg'])$$

where  $[g] = gH$ , commutes with the  ${}_cG$ -action

$$(1.3) \quad M_o^H \times G/H \times {}_cG \rightarrow M_o^H \times G/H, ((x, [g']), [n]) \rightarrow (n^{-1}x, [g'n])$$

where  $[n] = nH \in N(H)/H = {}_cG$ . Moreover, the  $G$ -map

$$(1.4) \quad M_o^H \times G/H \rightarrow M_o, (x, [g']) \rightarrow g'x$$

induces a  $G$ -diffeomorphism

$$(1.5) \quad M_o^H \times_{{}_cG} G/H := (M_o^H \times G/H)/{}_cG \rightarrow M_o$$

identifying  $M_o \rightarrow M_o/G$  with the  $G/H$ -bundle associated with the principal bundle (1.1).

The construction of the core of  $M$  (called the reduction of  $M$  in [16]) and later of the (partial) resolution of  $M$ , are natural extensions of the well known facts outlined above.

We refer to the closure,  ${}_cM := \text{cl}(M_o^H)$  of  $M_o^H$  in  $M$ , as the *core* of  $M$ . Clearly, each of the sets  $M_o^H \subset {}_cM \subset M^H$  are invariant under the  ${}_cG$ -action, and in general each inclusion is strict (cf. e.g. [16], for an example where  ${}_cM \neq M^H$ ). Our first objective is to analyze the structure of  ${}_cM$ .

The following possibly well known simple but very useful fact can be found in Kleiners thesis [12].

LEMMA 1.1. *Let  $c : [0, \ell] \rightarrow M$  be a minimal geodesic between the orbits  $Gc(0)$  and  $Gc(\ell)$ . Then  $G_{c(t)} = G_c$  for all  $t \in (0, \ell)$  and  $G_{c(0)} \supset G_c \subset G_{c(\ell)}$ .*

This together with the slice theorem can be used to give simple geometric proofs of all basic facts about compact transformation groups. Here we will use it in the proof of

**PROPOSITION 1.2.** *The core  ${}_cM$  is a smooth submanifold of  $M$ . In fact,  ${}_cM$  is the disjoint union of those components,  $F$  of  $M^H$  such that  $F \cap M_o \neq \emptyset$ . (For these components,  $\dim F = \dim {}_cG + \dim M/G$ .)*

*Proof.* Since  $(G/H)^H \simeq {}_cG$  and hence  $M_o^H$  has only finitely many components (all diffeomorphic), the inclusion  ${}_cM \subset \cup F$ ,  $F$  component of  $M^H$  with  $F \cap M_o^H \neq \emptyset$ , is obvious. Now let  $x \in F$ , with  $F$  as above. Choose  $y \in \text{cl}(F \cap M_o^H) \subset {}_cM$  closest within  $F$  to  $x$ . We will show that  $x = y$ . First note that since  $F \cap M_o^H$  is open in  $F$  and contains components of  ${}_cG$ -orbits in  $M_o^H$  arbitrarily close to  $y$ , we can find unit tangent vectors  $v \in T_y F$  such that by Lemma 1.1,  $\exp(tu) \in F \cap M_o^H$  for all  $u$  close to  $v$  and all small positive  $t$ , and hence all small  $t \neq 0$ . Now suppose  $x \neq y$  and let  $\gamma$  be a minimal geodesic in  $F$  from  $y$  to  $x$ . By assumption, all points of  $\gamma$  except  $y$  are in  $F - M_o^H$ . However, from the above, there are minimal geodesics  $c$  in  $F$  emanating from  $y$  all of whose points except  $y$  are in  $M_o^H$ , and such that  $c$  makes an angle less than  $\pi/2$  with  $\gamma$ . This contradicts the choice of  $y$ , and hence  $x = y$ .  $\square$

Our next goal is to determine the regular part of the core action  ${}_cG \times {}_cM \rightarrow {}_cM$  and its orbit space. We need the following

**LEMMA 1.3.** *Suppose  $G$  acts isometrically on the unit  $n$ -sphere,  $S$  with principal isotropy group  $H$ . Then  $N(H)/H \neq \{1\}$ .*

*Proof.* Assume without loss of generality that  $S^G = \emptyset$  (otherwise look at  $(S^G)^\perp$ ). A simple convexity argument shows that in this case  $\text{diam} S/G \leq \pi/2$ . If  $H = \{1\}$ , there is nothing to show. Now suppose  $H \neq \{1\}$  and  $N(H)/H = {}_cG = \{1\}$ . If  $G$  acts transitively,  $S = G/H$  we have  $S^H \simeq (G/H)^H \simeq {}_cG$  which is impossible since  $S^H$  is a subsphere. If  $G$  does not act transitively, consider  ${}_cS \supset (S_o)^H$  which in this case is a connected subsphere of  $S$ . The assumption  ${}_cG = \{1\}$  implies that  $\text{dist}(x, y) = \text{dist}({}_cGx, {}_cGy) = \text{dist}(Gx, Gy)$  for all  $x, y \in (S_o)^H$ . Since  $(S_o)^H$  is dense in  ${}_cS$ , and  $(S_o)^H \simeq (S_o)^H/{}_cG \simeq S_o/G$  is dense in  $S/G$  we get a contradiction from  $\text{diam } {}_cS = \pi$  and  $\text{diam} S/G \leq \pi/2$ .  $\square$

We are now ready to prove

**PROPOSITION 1.4.** *The inclusion  ${}_cM \subset M$  induces an isometry  ${}_cM/{}_cG \simeq M/G$  and  $({}_cM)_o = (M_o)^H$ . In particular,  ${}_cGx = Gx \cap {}_cM$  and  $({}_cG)_x = {}_c(G_x)$  for all  $x \in {}_cM$ .*

*Proof.* Clearly  $(M_o)^H \subset ({}_cM)_o$ . To prove the opposite inclusion let  $x \in {}_cM - (M_o)^H$ . We need to see that  $({}_cG)_x \neq \{1\}$ . However,  $({}_cG)_x = (N(H) \cap G_x)/H$ , and  $G_x$  acts on the normal sphere,  $S_x^\perp$  to the orbit  $Gx$  with principal isotropy group  $H$ . Thus  $({}_cG)_x$  can also be viewed as the core group  ${}_c(G_x)$  for the  $G_x$ -action on  $S_x^\perp$  and the claim follows from Lemma 1.3. Since  ${}_cM \subset M \rightarrow M/G$  is clearly surjective, and  $({}_cM)_o/{}_cG = (M_o)^H/{}_cG \simeq M_o/G$  is an isometry, the extension  ${}_cM/{}_cG \rightarrow M/G$  is an isometry as well.  $\square$

The following is a simple (and possibly well known) observation based on the construction above.

**THEOREM 1.5.** *Let  $M$  be a  $G$ -manifold with principal isotropy group  $H$ . If the core group  ${}_cG = N(H)/H$  is trivial, then all orbits are principal and  $M$  is  $G$ -equivalent to  ${}_cM \times G/H$ .*

*Proof.* Since  ${}_cG = \{1\}$ ,  $({}_cM)_o = {}_cM$ . But then  $M/G \simeq {}_cM/{}_cG = ({}_cM)_o/{}_cG = (M_o)^H/{}_cG = M_o/G$ , i.e.,  $M_o = M$ . Moreover,  $M = M_o \rightarrow M_o/G = M/G$  is a bundle with fiber  $G/H$  and trivial principal bundle  $M^H = (M_o)^H \xrightarrow{\sim} (M_o)^H/{}_cG = M^H \simeq M/G$ .  $\square$

This suggests an extension of Lemma 1.3 to manifolds of positive curvature:

**THEOREM 1.6.** *Let  $M$  be a closed manifold of positive curvature and  $G$  a compact group of isometries on  $M$ . If  $H \subset G$  is a principal isotropy group, then  $N(H)/H \neq \{1\}$  unless  $M = G/H$ .*

*Proof.* If  $N(H)/H = \{1\}$  we know from Theorem 1.5 that  $M$  is  $G$ -equivalent to  ${}_cM \times G/H$ . Moreover, the projection  $M \simeq {}_cM \times G/H \rightarrow {}_cM \simeq M/G$  is a flat Riemannian submersion, i.e., has trivial integrability tensor ( $G$ -translates of  ${}_cM$  are integral submanifolds of the horizontal distribution). Since a flat Riemannian foliation in a non-negatively curved manifold locally splits isometrically by [19, Theorem 1.3], this is impossible in positive curvature unless  ${}_cM$  is a point, i.e.,  $M = G/H$  is homogeneous.  $\square$

For calculations of  $N(H)/H$  when  $G$  is 1-connected, see [15], where it is called the generalized Weyl group. The description of  $G$ -manifolds whose core group  ${}_cG = N(H)/H$  is finite is considerably more complicated than Theorem 1.6. These are, however, special cases of so-called polar manifolds, see [14]. For a detailed analysis of polar manifolds we refer to [11]. Here, however, we point out that the arguments of Theorem 1.6 above can be pushed to yield the following extension:

**THEOREM 1.7.** *Let  $M$  be a closed manifold of positive curvature, and  $G$  a compact Lie group of isometries on  $M$ , with principal isotropy group  $H \subset G$ . If the core group  ${}_cG = N(H)/H$  is finite then either*

- (a)  $M = G/H$ , or
- (b) *There are singular orbits  $Gx$  on  $M$ , i.e., there are points  $x \in M - M_o$  with  $\dim G_x > \dim H$ .*

*Proof.* Suppose  $\dim G_x = \dim H$  for all  $x \in M$  and that  ${}_cG$  is finite. The first assumption implies that the  $G$ -orbits on  $M$  define a Riemannian foliation of  $M$ . The second assumption implies that this foliation is flat, i.e., has integrable horizontal distribution as in the proof of Theorem 1.6. As in Theorem 1.6 we get from [19] that  ${}_cM$  must be a point if  $M$  has positive curvature, hence  $M = G/H$ .  $\square$

The following immediate corollary played an important role in [10, Theorem B].

**COROLLARY 1.8 (Fixed point lemma).** *Let  $M$  be a closed manifold of positive curvature and  $G$  a compact connected Lie group of isometries on  $M$ . If the identity component  $H_o$  of the principal isotropy group  $H$  is a non-trivial maximal connected subgroup of  $G$ , then either*

- (a)  $M = G/H$ , or
- (b)  $M^G \neq \emptyset$ .

*Proof.* Since  $H_o$  is maximal we see that  $N(H)/H$  is finite. Otherwise,  $\dim N(H) = \dim G$  and hence  $M^H = M$ .  $\square$

We conclude this section by pointing out that part of the essence of the core group and manifold is, that it reduces many general questions about group actions to those that have trivial principal isotropy group. Some of the core constructions generalize to other types of "reductions" when replacing  $(M_o)^H$  by  $(M_o)^L$  for subgroups  $L \subset H$ . Among all these reductions, the core,  ${}_cM$  is the one reduced the most. This is the reason for choosing the word "core" for this most basic reduction. We will not pursue the more general reductions further in this note.

## 2. THE CORE-RESOLUTION CONSTRUCTION

With the construction of  $M_o$  in (1.5) as guideline, we will construct a new  $G$ -manifold,  ${}_cM$  which maps onto  $M$  but is less singular as a  $G$ -manifold, i.e., has smaller isotropy groups.

The  $G$ -action on  $M$  induces a smooth surjective map

$$(2.1) \quad F : {}_cM \times G/H \rightarrow M, (x, [g']) \rightarrow g'x$$

extending the map in (1.4). This is  $G$ -equivariant when  $G$  acts trivially on the  ${}_cM$ -factor, and by left translations on  $G/H$ . Moreover, it is  ${}_cG$ -invariant relative to the obvious  ${}_cG$ -extension to  ${}_cM \times G/H$  of (1.3). Thus  $F$  induces a surjective  $G$ -equivariant map

$$(2.2) \quad f : ({}_cM \times G/H)/{}_cG \rightarrow M, {}_cG(x, [g']) \rightarrow g'x.$$

Since the natural  ${}_cG = N(H)/H$ -action

$$(2.3) \quad G/H \times {}_cG \rightarrow G/H, ([g'], [n]) \rightarrow [g'n]$$

is free,  ${}^rM := ({}_cM \times G/H)/{}_cG = G/H \times_{{}_cG} {}_cM$  is a smooth manifold. We will refer to  ${}^rM$  as the (core-) *resolution* of  $M$ . Note that we can view  ${}^rM$  as a bundle over  $(G/H)/{}_cG = G/N(H)$  with fiber  ${}_cM$  associated to the principal  ${}_cG$ -bundle  $G/H \rightarrow G/N(H)$ . This bundle map

$$(2.4) \quad {}^rM \rightarrow G/N(H), (x, [g'])_{{}_cG} \rightarrow [g']_{{}_cG} =: (g'),$$

where  $(g') = g'N(H)$ , is clearly  $G$ -equivariant. We will now analyze the  $G$ -manifold  ${}^rM$  and the map  $f$  in (??) in more detail.

**THEOREM 2.1.** *Let  $M$  be a  $G$ -manifold with resolution  $f : {}^rM \rightarrow M$  as in (??), and  $F : {}_cM \times G/H \rightarrow M$  as in (2.1). Then*

- (1)  $f : {}^rM_o \rightarrow M_o$  is a  $G$ -diffeomorphism
- (2)  $f$  restricts to a  ${}_cG$ -diffeomorphism between the cores of  ${}^rM$  and of  $M$
- (3)  $f/G : {}^rM/G \rightarrow M/G$  is a homeomorphism.
- (4)  $G_{(x,[1])_{{}_cG}} = N(H) \cap G_x = N^{G_x}(H)$ ,  $(x, [1]) \in {}_cM \times G/H$ .
- (5)  $DF_{(x,[1])} : T_x({}_cM) \times T_{[1]}G/H \rightarrow T_xM$  is surjective if and only if  $(*)$   $T_x({}_cM) + T_xGx = T_xM$ .
- (6) The condition  $(*)$  is equivalent to  $G_x \subset N(H)$  as well as to  $(G_x)_o = (N(H) \cap G_x)_o$ , and  $f : {}^rM \rightarrow M$  is a  $G$ -diffeomorphism if this condition holds for all  $x \in {}_cM$ .

*Proof.* We prove (4) first: Let  $x \in {}_cM \subset M$ . Then

$$\begin{aligned} G_{(x,[1])_{{}_cG}} &= \{g \mid g(x, [1])_{{}_cG} = (x, [g])_{{}_cG} = (x, [1])_{{}_cG}\} \\ &= \{g \mid \exists n \in N(H) : (x, [g]) = (n^{-1}x, [n])\} \\ &= \{g \mid \exists n \in N(H) \cap G_x : gH = nH\} \\ &= \{g \mid \exists n \in N(H) \cap G_x : g \in nH\} \\ &= N(H) \cap G_x = N^{G_x}(H). \end{aligned}$$

This together with Lemma 1.3 (cf. also Proposition 1.4) shows that for  $x \in {}_cM$ ,  $(x, [1])_{{}_cG} \in {}^rM_o$  if and only if  $x \in {}_cM_o = M_o^H$ . In other words  ${}^rM_o = (M_o^H \times G/H)/{}_cG \simeq M_o$ . As for (2) first note that  $({}^rM_o)^H \simeq (M_o^H \times G/H)/{}_cG \simeq M_o^H$  and  $\text{cl}({}^rM_o)^H = ({}_cM \times N(H)/H)/{}_cG \simeq {}_cM$ . In particular,  $f$  restricts to a  ${}_cG$ -diffeomorphism from the core of  ${}^rM$  to the core of  $M$ . We get (3) as an immediate consequence of (2) and Proposition 1.4. Now, for  $x \in {}_cM$  and  $v \in T_x({}_cM)$ ,  $DF_{(x,[1])}(v, 0) = v \in T_x({}_cM) \subset T_xM$ . Moreover, for  $X \in T_{[1]}G/H \simeq \mathfrak{h}^\perp$  we have  $DF_{(x,[1])}(0, X) = X^*(x)$ , where  $X^*$  denotes the action field on  $M$  corresponding to  $X \in \mathfrak{g} \simeq T_1G$ . In particular,  $DF(T_{(x,[1])_{{}_cG}} {}_cM \times G/H) = T_x({}_cM) + T_xGx$  and (5) is proved. Since  $T_x({}_cM) \cap T_xGx = (T_xGx)^H$  and its complement in  $T_xGx$  is perpendicular to  $T_x({}_cM)$ , we see that the condition  $(*)$  in (5) is equivalent to the condition  $T_xGx^\perp \subset T_x({}_cM)$ . This on the other

hand says that  $H$  acts trivially on  $T_x Gx^\perp$ , i.e.,  $H$  is normal in  $G_x$ . The condition (\*) is also equivalent to the condition

$$\dim_c M + (\dim Gx - \dim_c Gx) = \dim M$$

by Proposition 1.4. The left hand side can be written as

$$\begin{aligned} \dim M_o^H + (\dim G - \dim G_x - \dim_c G + \dim_c G_x) = \\ \dim M/G + \dim_c G + (\dim G - \dim G_x - \dim_c G + \dim G_x \cap N(H) - \dim H) = \\ \dim M + \dim G_x \cap N(H) - \dim G_x, \text{ and } \dim G_x \cap N(H) = \dim G_x \end{aligned}$$

if and only if  $(G_x)_o = (G_x \cap N(H))_o$ . By  $G$ -equivariance,  $F$  and hence  $f$  is a submersion if and only if  $DF_{(x,[1])}$  is surjective for all  $x \in {}_c M$ . In this case,  $f$  is a covering map and thus a diffeomorphism by (1). This completes the proof of (6).  $\square$

*Remark 2.2.* From this theorem we see in particular that  ${}^r M/G = M/G$ . Moreover,  ${}^r M = M$  if and only if  $G_x \subset N(H)$  (or equivalently  $(G_x)_o = (N(H) \cap G_x)_o$ ) for all  $x \in {}_c M$  and hence  ${}^r({}^r M) = {}^r M$ . Among all  $G$ -manifolds with the same principal isotropy group and the same core  $({}_c M, {}_c G)$ , their common resolution is the least singular  $G$ -manifold. Also if  $M_1$  and  $M_2$  are two  $n$ -dimensional  $G$ -manifolds with  ${}^r M_1 = {}^r M_2$  as  $G$ -manifolds, then  ${}_c M_1 = {}_c M_2$  as  ${}_c G$ -manifolds. Note however, that  $({}_c M_1, {}_c G) = ({}_c M_2, {}_c G)$  does not imply that  $({}^r M_1, G) = ({}^r M_2, G)$  as the following example shows.

*Example 2.3.* If  $M = G/H$  is homogeneous, clearly  ${}^r M = M$ . In this case, the core  ${}_c M = (G/H)^H = N(H)/H \simeq {}_c G$  acting through right translations on itself. In particular if  $M_i = G/H_i, i = 1, 2$  and  $N(H_1)/H_1 = N(H_2)/H_2$  then  $({}_c M_1, {}_c G) = ({}_c M_2, {}_c G)$  but  ${}^r M_1 = M_1 \neq M_2 = {}^r M_2$  if  $G/H_1 \neq G/H_2$ . To be explicit, such examples can be found among the Aloff-Wallach examples [1],  $M_{p,q} = SU(3)/S_{p,q}^1$ . When  $(p, q) \neq (1, 1)$  and  $\gcd(p, q) = 1$ , then  $N(S_{p,q}^1)/S_{p,q}^1 = S^1$  (cf. e.g. [15]).

**COROLLARY 2.4.** *If  $M$  is a  $G$ -manifold without singular orbits then  ${}^r M = M$ .*

*Proof.* By assumption  $\dim G_x = \dim H$  for all  $x \in M$ . But for  $x \in {}_c M$  we have  $H \subset N(H) \cap G_x$  and hence  $H_o \subset (N(H) \cap G_x)_o \subset N(H)_o \cap (G_x)_o \subset (G_x)_o = H_o$ , and the claim follows from Theorem (2.1).  $\square$

For the following cf. also the example  $(M, G) = (S^n, SO(n))$  where  $S^n = \Sigma S^{n-1} = \Sigma SO(n)/SO(n-1)$ , and Theorem 1.5.

**COROLLARY 2.5.** *Let  $M$  be a 1-connected  $G$ -manifold with principal isotropy group  $H$ . If there are no singular orbits in  $M$ , and  $N(H)/H$  is finite, then all orbits are principal.*

*Proof.* From Corollary 2.4 and  ${}_c G = N(H)/H$  finite we see that  ${}_c M \times G/H \rightarrow {}^r M = M$  is a finite cover. Since  $M$  is simply connected this is a trivial  ${}_c G$ -bundle, i.e.,  ${}_c M \simeq {}_c G \times N$ , where  $N$  is a connected component of  ${}_c M$ , and  $M \simeq ({}_c G \times N) \times_{{}_c G} G/H \simeq N \times G/H$  as  $G$ -manifolds.  $\square$

Let us now turn to metric properties of the core resolution. - If  $M$  is a Riemannian  $G$ -manifold, it is natural to equip the totally geodesic core  ${}_c M$  with the induced Riemannian metric. It is also natural to endow  $G/H$  with a metric induced from a biinvariant on  $G$ . In this case, the product metric on  ${}_c M \times G/H$  will be invariant under both the  $G$ -action and the  ${}_c G$ -action. By the Gray-O'Neill curvature submersion formula (cf. [13] or [8]), we get the following interesting fact:

**PROPOSITION 2.6.** *The resolution  ${}^r M$  of any Riemannian  $G$ -manifold  $M$  with sectional curvature,  $\sec M \geq k, k \leq 0$  supports a  $G$ -invariant Riemannian metric with  $\sec {}^r M \geq k$ .*

*Remark 2.7.* This is particularly interesting for  $k = 0$ . For example, this provides a new proof that e.g.  $CP^n \# -CP^n$  has a metric with non-negative curvature, first proved in [6]. The point is that  $CP^n \# -CP^n$  is the resolution,  ${}^rS^{2n}$  of the  $U(n)$ -manifold  $S^{2n}$ , where the  $U(n)$ -action is the suspension of the standard action on  $S^{2n-1} = U(n)/U(n-1)$ .

From Corollary 2.4 and the description  ${}_cM \rightarrow {}^rM = {}_cM \times_{{}_cG} G/H \rightarrow G/N(H)$  it is tempting to make the following conjecture.

CONJECTURE. *Let  $M$  be a positively curved  $G$ -manifold with principal isotropy group  $H \neq \{1\}$ . Then either*

- (a)  $M = G/H$ , or
- (b)  ${}^rM \neq M$ . In particular there are singular  $G$ -orbits in  $M$ .

Note that the assumption  $H \neq \{1\}$  is necessary since there are many linear almost free actions of  $S^1$  and of  $S^3$  on spheres (cf. e.g. [9]). These together with the transitive actions on spheres all have  ${}^rS = S$ . Note that if the above conjecture is correct, then any (non finite) isometric  $G$ -action on a positively curved manifold,  $M$  has singular orbits, unless the action is transitive, or  $G$  acts almost freely on  $M$ , in which case  $\text{rk}G = 1$  and  $\dim M$  is odd.

Another interesting question about the resolution construction is whether it preserves the topological dichotomy into elliptic and hyperbolic types [7]. From its very construction this hinges on the question of whether the core can change type or not.

PROBLEM 2.8. *Is  ${}^rM$ , or equivalently  ${}_cM$  elliptic if and only if  $M$  is ?*

In conclusion we also point out that "intermediate" resolutions can be constructed by replacing the core by other reductions.

## REFERENCES

- [1] S. Aloff and N.R. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. **81**, (1975), 93–97.
- [2] A. Back and W.Y. Hsiang, *Equivariant geometry and Kervaire spheres*, Trans. Amer. Math. Soc. **304** (1987), 207–227.
- [3] L. Berard-Bergery, *Les variétés riemanniennes homogènes simplement connexes de dimension impaire à courbure strictement positive*, J. Math. Pures Appl. (9) **55**, (1976), 47–67.
- [4] M. Berger, *Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive*, Ann. Scuola Norm. Sup. Pisa (3) **15** (1961), 179–246.
- [5] G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972, Pure and Applied Mathematics, Vol. 46.
- [6] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Differential Geometry **8** (1973), 623–628.
- [7] Y. Félix and S. Halperin and J.-C. Thomas, *The homotopy Lie algebra for finite complexes*, Inst. Hautes Études Sci. Publ. Math. **56**, (1982), 387–410.
- [8] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16** (1967), 715–737.
- [9] D. Gromoll and K. Grove, *The low-dimensional metric foliations of Euclidean spheres*, J. Differential Geom. **28**, (1988), 143–156.
- [10] K. Grove and C. Searle, *Differential topological restrictions by curvature and symmetry*, J. Differential Geom. **47**, (1997), 530–559.
- [11] K. Grove and W. Ziller, *Characterizations of Polar Actions*, in preparation.
- [12] B. Kleiner, *Thesis*, Berkley (1990).
- [13] B. O’Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
- [14] Palais, R. and Terng, C.-L.: *A general theory of canonical form*, Trans. Amer. Math. Soc., **300** (1987), 771–789.
- [15] K. Shankar, *Isometry groups of homogeneous, positively curved manifolds*, Diff. Geom. and Appl., to appear.
- [16] T. Skjelbred and E. Straume, *On the reduction principle for compact transformation groups*, preprint (1993).

- [17] E. Straume, *Compact connected Lie transformation groups on spheres with low cohomogeneity. I,II*, Mem. Amer. Math. Soc. **119** (1996), no. 569 ; **125** (1997), no. 595.
- [18] N R. Wallach, *Compact homogeneous Riemannian manifolds with strictly positive curvature*, Ann. of Math. (2), **96**, (1972), 277–295.
- [19] G. Walschap, *Metric foliations and curvature*, J. Geom. Anal. **2**, (1992), 373–381.
- [20] A.G. Wasserman, *Simplifying group actions*, Topology Appl. **75**, (1997), 13–31.

UNIVERSITY OF MARYLAND, COLLEGE PARK , MD 20742

*E-mail address:* `kng@math.umd.edu`

INSTITUTO DE MATEMATICAS - UNAM, CUERNAVACA, MEXICO

*E-mail address:* `csearle@matcuer.unam.mx`